Solving Systems of Linear Equations

1 Solving Systems of Linear Equations

Recall the three Elementary Row operations (ERO'S)

1. Swap two rows

2. Multiply a row by a nonzero number

3. Add/subtract a multiple of one row to/from another

A Solving Systems of Linear Equations can have one of three outcomes, namely:

1. A Unique Solution - This means the planes meet at a point

2. No Solution - This means the planes are parallel

3. Infinitely Many Solutions - This means the planes meet along a line

1.1 Unique Solution

$$
\begin{aligned}
x + 2y + z &= 3 \\
3x - y - 3z &= -1 \\
2x + 3y + z &= 4
\end{aligned}
$$

Step 1: Use the coefficients in the above system of equations to generate the 3×4 augmented matrix

$$
A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix}.
$$

Step 2: Ensure that the first non zero entry in the first row is a 1 and add or subtract multiples of rows to get 0s in the rest of the column.

$$
\begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix} \xrightarrow[\substack{R_3 - 2R_1}]{\substack{R_2 - 3R_1}} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{pmatrix}
$$

Step 3: Now, ensure the first non-zero entry in the second row is a 1. The easiest way is here is to swap row 2 with row 3 and multiply by -1. When you have a 1 is this position use it to get os in the rest of that column.

$$\xrightarrow[R_2\times(-1)]{R_2\leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -7 & -6 & -10 \end{pmatrix} \xrightarrow[R_3+7\times R_2]{R_1-2\times R_2} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Step 4: Finally, ensure that the first non zero entry in the third row is a 1 and use it to get zeros in the rest of that column.

$$\xrightarrow[R_2+R_3]{R_2-R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

The matrix is now in *Reduced Row Echelon Form.* The first non-zeros entry in each row is a 1 and there area zeros above and below this entry.

Step 5: Now, by doing the reverse of Step 1 we obtain:

$$\begin{aligned} x &= 3 \\ y &= -2 \\ z &= 4 \end{aligned}$$

This system has a unique solution; specific values for x, y and z that can be thoughts of as a point.

1.2 No Solution

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{pmatrix}.$$

Try solving this in the same manner as before and see what happens!

1.3 Infinitely Many Solutions

$$A = \begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{pmatrix}.$$

This time we have more than one solution, in fact we have infinitely many solutions.
Try it now and see how you get on.
You may want to consult the MSC worksheet on Solving Matrix Equations (Free Variables).